tive modulus; $\alpha$, $\beta$, $\gamma$, constants characterizing the meniscus; $\varkappa$, strain rate; $\eta^* = \sigma/\varkappa$; c, m, constants; T, temperature; $t^*$, time of passage of a fixed section from z = R to the drum; $L_1$ and $L_2$, lengths of the stream being stretched and the falling stream; $F_{fr}$, friction; q, flow rate.

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# THE SPREADING OF A NON-NEWTONIAN LIQUID OVER A HORIZONTAL PLANE WITH INTENSIVE HEAT-TRANSFER AND MASS-TRANSFER PROCESSES ON THE SURFACE OF THE LAYER

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We investigate the flow of a layer of highly viscous non-Newtonian liquid over a horizontal plane, accompanied by intensive heat-transfer and mass-transfer processes.

We consider the behavior of a layer of viscoelastic liquid with a free surface on a solid horizontal plane. The coordinate system is so chosen that the axes OX and OY lie in this plane and the axis OZ is directed upward. In what follows, we shall consider the behavior of large liquid masses, and therefore we shall disregard capillary forces. For a highly viscous liquid the hydrodynamic problem is simplified. In the first place, the Reynolds numbers are small and the inertial terms may be neglected in the equations of motion. In the second place, the characteristic time scale of the flow is much longer than the relaxation time of the liquid (small Debora numbers), and the rheological equations of a nonlinearly viscoelastic liquid reduce to the rheological differential equation [1, 2] that is valid for slow flows:

$$\mathbf{T} = \eta \mathbf{A}_1 + \beta \mathbf{A}_1^2 + \nu \mathbf{A}_2. \tag{1}$$

Here T is the excess-stress tensor; $\mathbf{A}_1 = \left.\frac{D\mathbf{C}_t(\tau)}{D\tau}\right|_{\tau=t}$; $\mathbf{A}_2 = \left.\frac{D^2\mathbf{C}_t(\tau)}{D\tau^2}\right|_{\tau=t}$ Rivlin–Eriksen tensors [1, 2]; $C_t(\tau)$, Cauchy deformation tensor [1, 2]; coefficients $\eta$, $\beta$, $\nu$ depend on the second

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invariant tensor of the deformation rates; and tensor $A_1$ coincides with double the deformation-rate tensor [1, 2]. However, even after these simplifications the problem of the flow of a liquid with a free surface remains highly complex, even for a numerical solution (see the survey [3]). In the case when the characteristic scale of the flow L is considerably larger than the thickness h of the layer, we can use a stronger simplification of the problem — we can disregard the variation of the flow characteristics along the layer in comparison with their variation across the layer. Such an approximation is widely used in the theories of films and the lubricant use of a non-Newtonian liquid (see, e.g., [4, 5]). In [6, 7] a similar approach was used for calculating the spreading of a Newtonian liquid mass over a horizontal plane under isothermal conditions. In many cases the flow of the liquid is accompanied by intensive heat-transfer and mass-transfer processes: evaporation of a volatile solvent or chemical reactions on the surface of the layer, heating of the liquid from above or below, etc. Changes in the composition and temperature of the liquid cause a change in its effective viscosity. Therefore, the equations of heat transfer and mass transfer must be added to the equations of motion.

Starting from the continuity equations, we have: $v_x \sim U$, $v_y \sim U$, $v_z \sim hU/L$, $h \ll L$, where U is the scale of the spreading rate. When $\lambda \ll L/U$, where $\lambda$ is the maximum relaxation time in the spectrum, the trajectories of the motion of the liquid particles for the time interval $t - \tau < \lambda$, are determined by the relations: $x_\tau = x_t - v_x(x_t, y_t, z_t, t)(t - \tau)$, $z_\tau = z_t$, $y_\tau = y_t - v_y(x_t, y_t, z_t, t)(t - \tau)$. If for such motion we calculate the deformation tensor $C_t(\tau)$ and then the Rivlin–Eriksen tensors, and take account of the fact that $\beta \sim \eta\lambda$, $\nu \sim \eta\lambda$, we obtain an estimate for the components of the excess stresses in (1). After this, we can estimate the various terms of the equations of motion. The relations of the inertial forces to the viscous forces is of the order of $\rho U h^2/\eta L$, where $\rho$ is the density of the liquid. The above estimates show that when

$$\frac{\rho U h^2}{\eta L},\ \frac{h}{L},\ \lambda\frac{U}{L} \ll 1,\ U \sim \frac{\rho g h^3}{\eta L}, \tag{2}$$

the equations of motion of the liquid layer and the equations of heat and mass transfer in it are simplified (P is the pressure):

$$\frac{\partial T_{xz}}{\partial z} = \frac{\partial P}{\partial x},\quad \frac{\partial T_{yz}}{\partial z} = \frac{\partial P}{\partial y},\quad \frac{\partial P}{\partial z} = -\rho g, \tag{3}$$

$$\rho c_p\left(\frac{\partial\theta}{\partial t} + v_x\frac{\partial\theta}{\partial x} + v_y\frac{\partial\theta}{\partial y} + v_z\frac{\partial\theta}{\partial z}\right) = \frac{\partial}{\partial z}\left(\Lambda\frac{\partial\theta}{\partial z}\right) + \Phi, \tag{4}$$

$$\frac{\partial c}{\partial t} + v_x\frac{\partial c}{\partial x} + v_y\frac{\partial c}{\partial y} + v_z\frac{\partial c}{\partial z} = \frac{\partial}{\partial z}\left(D\frac{\partial c}{\partial z}\right). \tag{5}$$

Here $\theta$ is the temperature; c, concentration; $\Phi = T_{xz}\frac{\partial v_x}{\partial z} + T_{yz}\frac{\partial v_y}{\partial z}$, dissipation functions; $\Lambda$, D, thermal conductivity and the diffusion coefficient; and $c_p$, specific heat capacity. Equations (3)-(5) are analogous to the equations used in the theory of films [4] and the theory of lubricant use [5] of nonlinearly viscous liquids. The above estimates show that if the condition (2) is satisfied, the elastic properties do not affect the motion of the liquid layer. We need to take into account only the variation of the shear viscosity as a function of the shear rate. To the equations (3) we should add the incompressibility condition

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0, \tag{6}$$

the boundary conditions on the free surface, which for (2) can be written in the form

$$T_{xz}|_{z=h} = T_{yz}|_{z=h} = 0,\quad P|_{z=h} = P_0, \tag{7}$$

$$\frac{\partial h}{\partial t} + v_x|_{z=h}\frac{\partial h}{\partial x} + v_y|_{z=h}\frac{\partial h}{\partial y} - v_z|_{z=h} = 0, \tag{8}$$

and the boundary conditions on the horizontal plane:

$$v_x|_{z=0} = v_y|_{z=0} = 0, \quad v_z|_{z=0} = -V(x, y, t). \tag{9}$$

Here h is the height of the free surface; $V = \sum_i V_i \delta(x - x_i)\,\delta(y - y_i)$, flow rate through the i-th opening; $\delta(\ )$, delta function; $x_i$, $y_i$, coordinates of the opening; and $P_0$, pressure above the liquid. Determining $v_z$ from (3), (9) and substituting this formula into (8), we obtain, after some simple calculations analogous to those given in [4], the equation for the height of the free surface:

$$\frac{\partial h}{\partial t} + \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + V = 0, \tag{10}$$

where $Q_x = \int_0^h (h - z)\frac{\partial v_x}{\partial z}\,dz$, $Q_y = \int_0^h (h - z)\frac{\partial v_y}{\partial z}\,dz$ are the liquid flows along the axes OX and OY. The distribution of the pressure and the shearing stresses in the layer can be found from Eqs. (3), (7):

$$P = P_0 + \rho g(h - z), \quad T_{xz} = \rho g(z - h)\frac{\partial h}{\partial x}, \quad T_{yz} = \rho g(z - h)\frac{\partial h}{\partial y}. \tag{11}$$

For many liquids the variation of the shear viscosity as a function of shear rate, temperature, and composition is determined by using the reduced quantities [8]:

$$\eta = \eta_0 a(\theta, c) f[\lambda_0 a(\theta, c) S^{1/2}], \quad S = 2\,\mathrm{tr}\,D^2, \tag{12}$$

where D is the deformation-rate tensor; $a(\theta, c)$, shear factor taking account of the variation in temperature and composition; $a = 1$ at the reduced temperature and composition; $\eta_0$, $\lambda_0$, initial non-Newtonian viscosity and the relaxation time at the reduced temperature and composition. From formula (12) it follows that for the flow under consideration

$$\frac{\partial v_x}{\partial z} = \frac{T_{xz}}{\eta_0 a}\psi, \quad \frac{\partial v_y}{\partial z} = \frac{T_{yz}}{\eta_0 a}\psi, \quad S = \left(\frac{\partial v_x}{\partial z}\right)^2 + \left(\frac{\partial v_y}{\partial z}\right)^2, \tag{13}$$

where $\psi = 1/f(\lambda_0 a S^{1/2})$, and the quantity $\lambda_0 a S^{1/2}$ is found from the expression

$$\frac{(T_{xz}^2 + T_{yz}^2)^{1/2}}{\eta_0/\lambda_0} = f(\lambda_0 a S^{1/2})\,\lambda_0 a S^{1/2}.$$

The relations (13) and (11) enable us to calculate the flows $Q_x$ and $Q_y$:

$$Q_x = -\frac{\rho g h^3}{\eta_0}\beta\frac{\partial h}{\partial x}, \quad Q_y = -\frac{\rho g h^3}{\eta_0}\beta\frac{\partial h}{\partial y}, \tag{14}$$

where

$$\beta = \int_0^1 \frac{(1-\zeta)^2}{a(\theta, c)}\,\psi\left[\frac{\rho g h}{\eta_0/\lambda_0}\sqrt{\left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2}\,(1-\zeta)\right]d\zeta, \quad \zeta = \frac{z}{h}.$$

Thus, the variation of the height of the liquid layer as a function of time is described by the nonlinear parabolic equation

$$\frac{\partial h}{\partial t} = \nabla\left(\frac{\rho g h^3}{\eta_0}\beta\nabla h\right) - \sum_i V_i \delta(x - x_i)\,\delta(y - y_i). \tag{15}$$

The runoff velocity in (15) usually depends only on the pressure on the plane. The relation $V_i = V_i[P_0 + \rho g h(x_i, y_i, t)]$ is determined by the hydraulic characteristics of the runoff structure. When the dimension of the region near the free surface, where the initial non-Newtonian viscosity of the liquid is reached, is so small that for all temperatures, compositions, and shear rates we may use a power law for the viscosity $f = (\lambda_0 a S^{1/2})^{n-1}$, then

$$\beta = \left\{\frac{\rho g h}{\eta_0/\lambda_0}\left[\left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2\right]^{1/2}\right\}^{\frac{1-n}{n}} \int\limits_0^1 \frac{(1-\zeta)^{1+\frac{1}{n}} d\zeta}{a(\theta,\ c)}. \tag{16}$$

Let us analyze in more detail the equations of heat and mass transfer (4), (5) for the flow under consideration. The dissipative heat generation heats the liquid in time L/U by an amount $\Delta\theta_d \simeq gh/c_p$, which usually does not exceed several tens of degrees. Therefore, the dissipation of energy may be disregarded in (4). The boundary conditions for Eqs. (4), (5) when z = h are determined by the physicochemical processes taking place on the free surface. The kinetic equations for these processes relate the temperature $0|_{z=h}$ and the composition $c|_{z=h}$ of the liquid to the heat flux $j_0$ and the mass flux $j_c$ removed from the layer. If the inequalities (2) are satisfied, then

$$-j_c = \left(D\frac{\partial c}{\partial z} + c\,\frac{\partial h}{\partial t}\right)_{z=h}, \quad -j_\theta = \left(\Lambda\,\frac{\partial\theta}{\partial t} + \rho c_p\theta\frac{\partial h}{\partial t}\right)_{z=h}. \tag{17}$$

The boundary conditions for a horizontal surface with z = 0 can be set up analogously. The nature of the influence exerted by the heat and mass transfer on the motion of the liquid is determined by the relation between the spreading time L/U and the times of the thermal process ($h^2/K$, where $K = \Lambda/\rho c_p$ is the thermal diffusivity) and the diffusion process ($h^2/D$). The same relations characterize the ratio of the molecular and convective transfers described in Eqs. (4), (5). When the spreading time is much smaller than the characteristic times of the thermal and diffusion processes

$$\frac{h^2}{K} \gg \frac{L}{U}, \quad \frac{h^2}{D} \gg \frac{L}{U}, \tag{18}$$

the effect of the heat and mass transfer on the surfaces of the layer is propagated only to a slight depth in the liquid, much less than the thickness of the liquid layer. When we calculate the distribution of temperature and concentration near the free surface, in (4), (5) we may set: $v_x \approx v_x|_{z=h},\ v_y \approx v_y|_{z=h},\ v_t \approx v_z|_{z=h}$. Near the free surface we shall seek the solution of Eqs. (4), (5) in the form: $\theta = \theta(\sigma, t),\ c = c(\sigma, t)$, where $\sigma = h - z$. Substituting these expressions into (4), (5) and taking account of (8), we find:

$$\rho c_p \frac{\partial\theta}{\partial t} = \frac{\partial}{\partial\sigma}\left(\Lambda\,\frac{\partial\theta}{\partial\sigma}\right), \quad \frac{\partial c}{\partial t} = \frac{\partial}{\partial\sigma}\left(D\,\frac{\partial c}{\partial\sigma}\right). \tag{19}$$

In the interior of the liquid volume, if the inequalities (18) are satisfied, we may disregard molecular transfer in Eqs. (4), (5). Thus, within the liquid layer the distribution of temperature and concentration is described by the equations

$$\begin{gathered}\frac{\partial\theta}{\partial t} + v_x\frac{\partial\theta}{\partial x} + v_y\frac{\partial\theta}{\partial y} + v_z\frac{\partial\theta}{\partial z} = 0,\\ \frac{\partial c}{\partial t} + v_x\frac{\partial c}{\partial x} + v_y\frac{\partial c}{\partial y} + v_z\frac{\partial c}{\partial z} = 0.\end{gathered} \tag{20}$$

If at the initial instant of time the entire liquid had a uniform temperature and composition: $\theta|_{t=0} = \theta_i$, $c|_{t=0} = c_i$, then, as can be seen from (20), when the liquid spreads, the temperature and composition do not change within the layer: $\theta = \theta_i$, $c = c_i$. Near the solid surface the following relations may be used for the field of velocities: $v_x \approx \gamma_x z,\ v_y \approx \gamma_y z,\ v_z \approx -\left(\frac{\partial\gamma_x}{\partial x} + \frac{\partial\gamma_y}{\partial y^2}\right)\frac{z^2}{2}$, where $\gamma_x$, $\gamma_y$ are the values of the velocity gradients on the horizontal surface. During the spreading time, changes in temperature and concentration take place in regions of thickness $\Delta_\theta \sim \sqrt{KL/U},\ \Delta_c \sim \sqrt{DL/U}$. If the conditions of (18) are satisfied, then $U\Delta_\theta^3/hLK \ll 1,\ U\Delta_c^3/hLD \ll 1$ and the contribution made by the convective terms to Eqs. (4), (5) near the solid surface is insignificant. Therefore, near the solid surface the temperature and concentration distributions are described by the equations

$$\rho c_p \frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z}\left(\Lambda \frac{\partial \theta}{\partial z}\right), \quad \frac{\partial c}{\partial t} = \frac{\partial}{\partial z}\left(D \frac{\partial c}{\partial z}\right). \tag{21}$$

For a uniform distribution of temperature and concentration at the initial instant of time, we must set up for Eqs. (19), (21) the following initial and boundary conditions, respectively:

$$\theta|_{t=0} = \theta_i, \ \theta|_{\sigma\to\infty} \to \theta_i; \ c|_{t=0} = c_i, \ c|_{\sigma\to\infty} \to c_i, \tag{22}$$

$$\theta|_{t=0} = \theta_i, \ \theta|_{z\to\infty} \to \theta_i; \ c|_{t=0} = c_i, \ c|_{z\to\infty} \to c_i. \tag{23}$$

The boundary conditions when $\sigma = 0$ for (19) and $z = 0$ for (21) are given on the free and solid surfaces. Our analysis shows that when the inequalities (18) are satisfied and the initial distributions of temperature and concentration are uniform throughout the liquid, the integral expression for the quantity $\beta$ can be broken up into three parts:

$$\beta \approx \int_0^1 \frac{(1-\zeta)^2}{a_i} \psi\left[\frac{\rho g h}{\eta_0/\lambda_0} \sqrt{\left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2}\,(1-\zeta)\right] d\zeta +$$

$$+ \frac{1}{h} \psi\left[\frac{\rho g h}{\eta_0/\lambda_0} \sqrt{\left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2}\right] \int_0^\infty \left[\frac{1}{a[\theta(z, t), c(z, t)]} - \frac{1}{a_i}\right] dz +$$

$$+ \frac{\psi(0)}{h^3} \int_0^\infty \sigma^2 \left[\frac{1}{a[\theta(\sigma, t), c(\sigma, t)]} - \frac{1}{a_i}\right] d\sigma, \quad a_i = a(\theta_i, c_i),$$

corresponding to the internal volume of the liquid and the thin layers near the solid and free surfaces. In an analogous manner, we can subdivide into three parts the integral (16) for the power-law model:

$$\int_0^1 \frac{(1-\zeta)^{1+\frac{1}{n}} d\zeta}{a(\theta, c)} \approx \frac{n}{(2n+1)a_i} + \frac{1}{h}\int_0^\infty \left[\frac{1}{a(\theta, c)} - \frac{1}{a_i}\right] dz + \frac{1}{h^{2+1/n}} \int_0^\infty \sigma^{1+1/n}\left(\frac{1}{a(\theta, c)} - \frac{1}{a_i}\right) d\sigma.$$

The resulting formulas for $\beta$ show the radically different ways in which the motion of the liquid layer is affected by the heat-exchange and mass-exchange processes on the solid surface and the free surface. When the spreading time is much longer than the characteristic times of the thermal and diffusion processes:

$$\frac{h^2}{K} \ll \frac{L}{U}, \quad \frac{h^2}{D} \ll \frac{L}{U}, \tag{24}$$

which is possible in sufficiently thin layers of liquid, we can disregard the convective terms in Eqs. (4), (5). In this case

$$\rho c_p \frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z}\left(\Lambda \frac{\partial \theta}{\partial z}\right), \quad \frac{\partial c}{\partial t} = \frac{\partial}{\partial z}\left(D \frac{\partial c}{\partial z}\right), \tag{25}$$

and, in view of the slow variation of the level of liquid in comparison with the rate of the processes of heat and mass exchange indicated in (24), when we calculate the concentration and temperature, the variation of h as a function of time in (25) may be disregarded. In the quasistationary approximation $\theta \approx \theta\left(\frac{z}{h}, t\right)$, $c \approx c\left(\frac{z}{h}, t\right)$. Therefore, in the integral for $\beta$ we have $a = a[\theta(\zeta, t), c(\zeta, t)]$. In particular, for the power-law model the integral in (16) is independent of h:

$$\frac{1}{a_{eq}(t)} = \int_0^1 \frac{(1-\zeta)^{1+\frac{1}{n}} d\zeta}{a[\theta(\zeta, t), c(\zeta, t)]}.$$

In the above-described limiting cases of a thermally thick layer (18) and a thermally thin layer (24), the heat-conduction equation (4) and the diffusion equation (5) can be solved analytically. As a result, the effect of the heat-exchange and mass-exchange processes on the motion of the liquid is determined only by the nature of the variation of the coefficient $\beta$ in (15) as a function of h and $|\nabla h|$. Under these conditions, the flow of the liquid layer is described by one quasilinear parabolic equation (15). Equations of this type occur in problems concerning the propagation of heat in media with thermal conductivities and a volumetric heat source which depend on temperature in a nonlinear manner, and also in problems concerning the filtration of nonlinearly viscous liquids. Thermal problems have been analyzed in detail in many studies carried out under the guidance of academician A. A. Samarskii at the M. V. Keldysh Applied Mechanics Institute of the Academy of Sciences of the USSR (see the surveys [9, 10]). Filtration problems have been described in [11, 12]. For a power-function rheological law (16), Eq. (15) with no sinks for the axially symmetric case and the plane case has a self-similar analytic solution, found in [13, 14]. It describes the spread of a liquid mass over a plane for sufficiently large values of time, when the motion of the liquid is independent of the initial shape of the layer. For the axially symmetric case this solution satisfies the integral condition that the liquid volume v remains constant, and for the plane case it satisfies the condition that the cross-sectional area s remains constant:

$$\int_0^\infty h 2\pi r dr = v, \quad \int_0^\infty h dx = s.$$

Using these self-similar solutions, we can illustrate the effect of the processes of heat and mass exchange on the way in which the liquid spreads by using three examples: a thermally thick layer of cold liquid heated either from above or from below, when the fluidity of the liquid may be neglected, $1/a_i \approx 0$, and also a thermally thin layer, when the effect of the heat and mass exchange is described by one characteristic $a_e(t)$. In the first and second cases the calculation of the temperature and concentration fields reduces to the traditional problems of heat and mass transfer, (19) or (21), for a semiinfinite layer with boundary conditions in the interior given by (22). The literature contains many studies on the calculation of these fields for the most diverse physicochemical phenomena on the surface: evaporation, combustion, dissolution, radiant and convective heating, etc. The formulas obtained in these studies for the distributions of temperature and concentration should be substituted into the expression for $\beta$. Disregarding the fluidity of the cold liquid and assuming that the distributions of temperature and concentration are independent of h, we find for (16) in the cases of heating from above and from below, respectively:

$$\beta = \frac{I_1(t)}{h^3}\left(\frac{\rho g}{\eta_0/\lambda_0}\left|\frac{\partial h}{\partial r}\right|\right)^{\frac{1}{n}-1}, \quad I_1(t) = \int_0^\infty \sigma^{1+\frac{1}{n}}\left(\frac{1}{a(\theta, c)} - \frac{1}{a_i}\right) d\sigma,$$

$$\beta = \frac{I_2(t)}{h^{2-1/n}}\left(\frac{\rho g}{\eta_0/\lambda_0}\left|\frac{\partial h}{\partial r}\right|\right)^{\frac{1}{n}-1}, \quad I_2(t) = \int_0^\infty \left(\frac{1}{a(\theta, c)} - \frac{1}{a_i}\right) dz.$$

In particular, when boundary conditions of the first kind can be given for the temperature and concentration on the free and solid surfaces, their distribution is described by simple known relations ($\theta_s$, $c_s$ are the values on the surface; $\delta$ is the distance from the surface):

$$\theta = \theta_i + (\theta_s - \theta_i)\,\mathrm{Erfc}\left(\frac{\delta}{2\sqrt{Kt}}\right), \quad c = c_i + (c_s - c_i)\,\mathrm{Erfc}\left(\frac{\delta}{2\sqrt{Dt}}\right).$$

In this case:

$$I_1(t) = (Kt)^{1+\frac{1}{2n}}\frac{1}{a_{e_1}}, \quad \frac{1}{a_{e_1}} = \int_0^\infty p^{1+\frac{1}{n}}\left(\frac{1}{a(p)} - \frac{1}{a_i}\right) dp,$$

$$I_2(t) = (Kt)^{1/2}\frac{1}{a_{e_2}}, \quad \frac{1}{a_{e_2}} = \int_0^\infty \left(\frac{1}{a(p)} - \frac{1}{a_i}\right) dp,$$

$$a(p) = a\left[\theta(p),\ c\left(\sqrt{\frac{K}{D}}\,p\right)\right], \quad p = \frac{\delta}{\sqrt{Kt}}.$$

For a thermally thin layer:

$$\beta = \frac{1}{h^{1-\frac{1}{n}}a_e(t)}\left(\frac{\rho g}{\eta_0/\lambda_0}\left|\frac{\partial h}{\partial r}\right|\right)^{\frac{1}{n}-1}.$$

As the formulas for the temperature and concentration distributions in the liquid required for calculating $a_e(t)$, we can use the results of many published calculations of heat and mass transfer in thin layers on a solid surface. In the simplest case of boundary conditions of the first kind, on the free and solid surfaces in the quasistationary approximation (see (24)) we have:

$$\theta \approx \theta_1(t)\left(1 - \frac{z}{h}\right) + \theta_2(t)\frac{z}{h}, \quad c \approx c_1(t)\left(1 - \frac{z}{h}\right) + c_2(t)\frac{z}{h},$$

where the subscripts 1 and 2 denote quantities on the free and solid surfaces. The spreading of the liquid under isothermal conditions is described by the formulas for a thermally thin layer when $a_e = 1$.

For the cases considered, the axially symmetrically self-similar solution of Eq. (15) without sinks is written in the form

$$h = \frac{Hf(\xi)}{\left[\tau\left(\frac{\rho g H}{\eta_0\lambda_0^{n-1}}\right)^{1/n}\right]^{\frac{2}{m}}}, \quad \xi = \frac{r}{H\left[\tau\left(\frac{\rho g H}{\eta_0\lambda_0^{n-1}}\right)^{1/n}\right]^{\frac{1}{m}}}.$$

Here $H = (v/2\pi)^{1/3}$, $\eta_0\lambda_0^{n-1}$ is the consistency of the liquid, and the function $f(\xi)$ is determined from the equations

$$\frac{d}{d\xi}\left[\xi\left|\frac{df}{d\xi}\right|^{1/n}\operatorname{sign}\left(\frac{df}{d\xi}\right)f^q\right] = -\frac{1}{m}\frac{d}{d\xi}(\xi^2 f), \quad \int_0^\infty \xi f\, d\xi = 1. \tag{26}$$

For a thermally thick layer heated from above and from below, respectively,

$$q = 0, \quad m = \frac{3}{n} - 1, \quad \tau = \int_0^t \frac{I_1(t)\,dt}{H^{2+1/n}},$$

$$q = 1 + \frac{1}{n}, \quad m = 1 + \frac{5}{n}, \quad \tau = \int_0^t \frac{I_2(t)\,dt}{H}.$$

For a thermally thin layer: $q = 2 + \frac{1}{n}$, $m = 3 + \frac{5}{n}$, $\tau = \int_0^t \frac{dt}{a_e(t)}$. In the case of heating of a thermally thick layer from above, the differential equation (26) has a solution satisfying the integral condition of conservation only for n < 3. This solution can be written as follows [B( ) is the beta function]:

$$f[1 < n < 3] = \left[\frac{(n+1)\left(\frac{3}{n} - 1\right)^n}{(n-1)(\xi_0^{n+1} + \xi^{n+1})}\right]^{\frac{1}{n-1}},$$

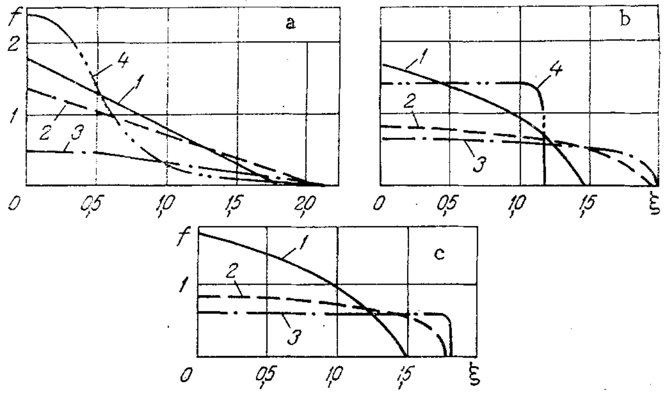

Fig. 1. Spread of a liquid heated from above (a) and from below (b) in a thermally thick layer and of a liquid in a thermally thin layer (c) for various values of n: a) 0 (1); 0.1 (2); 1.1 (3); 2.2 (4); b) 0 (1); 1 (2); 3 (3); 50 (4); c) 0 (1); 1 (2); 20 (3).

$$\xi_0^{\frac{3-n}{n-1}} = \left[\frac{(n+1)\left(\frac{3}{n}-1\right)^n}{n-1}\right]^{\frac{1}{n-1}} \frac{B\left(\frac{2}{n+1},\ \frac{3-n}{(n+1)(n-1)}\right)}{n+1},$$

$$f\,[n=1] = \frac{1}{2}\exp\left(-\xi^2/4\right),$$

$$f\,[n<1] = \left[\frac{(1-n)\left(\xi_0^{n+1}-\xi^{n+1}\right)}{(1+n)\left(\frac{3}{n}-1\right)^n}\right]^{\frac{1}{1-n}},$$

$$\xi_0^{\frac{3-n}{1-n}} = \left[\frac{(1+n)\left(\frac{3}{n}-1\right)^n}{1-n}\right]^{\frac{1}{1-n}} \frac{n+1}{B\left(\frac{2}{n+1},\ \frac{2-n}{1-n}\right)}.$$

For a thermally thick layer heated from below, for all values of n:

$$f = \left[\frac{2\left(\xi_0^{n+1}-\xi^{n+1}\right)}{(n+1)\left(1+\frac{5}{n}\right)^n}\right]^{1/2},$$

$$\xi_0^{\frac{n+5}{2}}\left[\frac{1}{2}(n+1)\left(1+\frac{5}{n}\right)^n\right]^{1/2} \frac{n+1}{B\left(\frac{2}{n+1},\ \frac{3}{2}\right)}.$$

For a thermally thin layer, for all values of n:

$$f = \left[\frac{(n+2)\left(\xi_0^{n+1}-\xi^{n+1}\right)}{(n+1)\left(3+\frac{5}{n}\right)^n}\right]^{\frac{1}{n+2}},$$

$$\xi_0^{\frac{3n+5}{n+2}} = \left[\frac{(n+1)\left(3+\frac{5}{n}\right)}{n+2}\right]^{1/2} \frac{n+1}{B\left(\frac{2}{n+1},\ \frac{n+3}{n+2}\right)}.$$

Figure 1 shows the graphs of the function f for various values of n. When n = 0 for a thermally thick layer heated from above, we have $f = 6^{1/3} - \xi$; when it is heated from below, and also in the case of a thermally thin layer, we have $f = [2(\xi_0 - \xi)]^{1/2}$, $\xi_0 = 15^{2/5}/2$. As $n \to \infty$, for a thermally thick layer heated from below we have f = 2 when $\xi < 1$, $\xi_0 = 1$, and for a thermally thin layer we have $f = 6^{1/3}/3$ when $\xi < 6^{1/3}$, $\xi_0 = 6^{1/3}$. As n → 3, for a thermally thick layer heated from above we have f → 0 when $\xi > 0$ and f → ∞ when $\xi = 0$. This shows that we cannot use the approximation of a thermally thick layer in the case of heating from above for $n \geqslant 3$. In this case the heating of the layer takes place more rapidly than the spreading.

The above results for the axially symmetric case can easily be carried over to the case of flow in one direction, which takes place when a long strip spreads over a surface. For isothermal spreading of a Newtonian liquid the formulas we have obtained coincide with the results of [6, 7].

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